On fairness in Bandwidth Allocation

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Abstract: For over a decade, the Nash bargaining solution (NBS) concept from cooperative game theory has been used in networks as a concept that allows one to share resources fairly. Due to its many appealing properties, it has recently been used for assigning bandwidth in a general topology network between applications that have linear utilities. In this paper, we use this concept for the bandwidth allocation between applications with general concave utilities. We study the impact of concavity on the allocation and present computational methods for obtaining fair allocations in a general topology, based on a dual Lagrangian approach and on Semi-Definite Programming.

Key-words: Nash Bargaining, bandwith allocation, fairness

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La notion d'équité dans l'allocation de bande passante

Résumé : Le concept de Nash Bargaining Solution (NBS), né de la théorie des jeux coopératifs, est utilisé depuis plus de dix ans dans les réseaux pour permettre le partage équitable des ressources. Grâce à ses propriétés intéressantes, il a récemment été utilisé dans des problèmes d'allocation de bande passante dans des réseaux aux topologies quelconques où se côtoient des applications aux fonctions d'utilité linéaires. Dans cet article, nous utilisons le NBS dans le cadre de l'allocation de bande passante entre des applications aux fonctions d'utilités concaves quelconques, étudions l'impact de la concavité sur l'allocation et présentons des méthodes calculatoires pour obtenir des allocations équitables dans une topologie générale, basées sur une approche de dual Lagrangien et de programmation semi-définie.

Mots-clés : équité, allocation de bande passante, Nash Bargaining

Contents

1 Introduction

Fair bandwidth assignment has been one of the important challenging areas of research and development in networks supporting elastic traffic. Indeed, Max-min fairness has been adopted by the ATM forum for the Available Bit Rate (ABR) service of ATM [1]. Although the max-min fairness has some optimality properties (Pareto optimality), it has been argued that it favors too much long connections and does not make efficient use of available bandwidth. In contrast, the concept of proportional fairness (of the throughput assignment) has been proposed by Kelly [11, 7], which gives rise to a more efficient solution in terms of network resources by providing more resources to shorter connections. An assignment is proportionally fair if any change in the distribution of the assigned rates would result in the sum of the proportional changes to be non-positive.

Although the object that is shared fairly seems to be a very specific one: the throughput, it is shown in [11, 7] that in fact, the starting point for obtaining (weighted) proportional fairness of the throughput can be a general (concave) utility function per connection; it is then shown that a global minimization of (weighted) sum of these utilities leads to a weighted proportional fair assignment of the throughput. As opposed to this approach, we wish to use a fairness concept which is defined directly in terms of the utilities of users rather than in terms of the throughputs they are assigned. Yet, as in weighted proportional fairness, it would be desirable to obtain this concept as the solution of a utility maximization problem, since it makes it possible to use recent algorithms for utility maximization in networks, along with decentralized implementations [10, 9, 12].

NBS is a natural framework that allows us to define and design fair assignment of bandwidth between applications with different concave utilities and has already been used in networking problems [14, 8]. It is characterized by a set of axioms that are appealing in defining fairness. As already recognized in [7] and later in [8], proportional fairness agrees with NBS in case that the object that is shared fairly is the throughput (and the minimum required rate is zero). We use NBS to study the fairness of an assignment where connection i has a concave utility over an interval $[MR_i, PR_i]$. It thus has a minimum rate requirement MR_i and does not need more than PR_i. Utility functions with similar features have been identified in [17] for representing some real time applications such as voice and video, and in the case that $MR_i = 0$, for elastic traffic.

We study in this paper the way the concavity of the utilities affect the bandwidth assignment according to NBS, as well as according to a generalized version of the proportional fairness (in which the utilities that correspond to different assignments, instead of the throughputs, are fairly allocated). Both notions are introduced in Sec. 2 and their properties are studied in Sec. 3. We then propose in Sec. 4 a quadratic approximation for the utility of each connection, which allows us to parameterize the degree of concavity of the utility function using a single parameter. We use this approximation to further analyze the impact of concavity of utilities on the resulting assignment. We then present in Sec. 5 a Lagrangian approach which allows us to implement a decentralized protocol for the bandwidth allocation. We finally present in Sec. 6 a novel alternative approach using Semi Definite Programming (SDP).

2 General problem

2.1 Utility function

The fairness problem which we consider is how to allocate bandwidth to connections beyond their minimum required bandwidth (MR). (We assume that if the minimum required bandwidth is not available then the connection is not accepted by the network.) The fairness issue is of interest only in the case when the utility of an application strictly increases when allocated more bandwidth than its *MR*. Connections with on/off utility functions (which characterize some applications with hard real-time requirements, [17]) are thus ignored in allocating extra bandwidth once they receive their *MR*.

Two kinds of applications are considered in [17] for which the fair allocation is relevant:

Elastic applications: Examples of such applications are file transfer or email. The typical utility function is concave increasing without a required minimum rate, see Fig. 1.

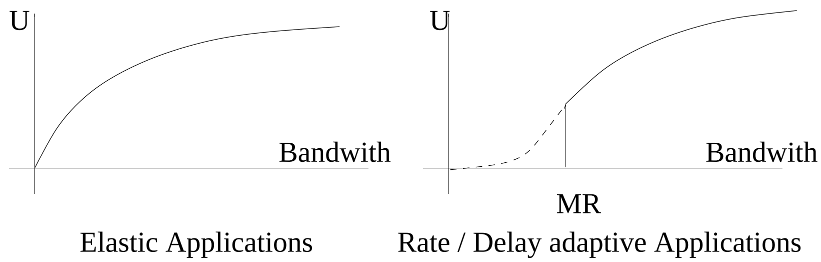

Figure 1: Utility function of elastic (left) and of rate-adaptive or delay-adaptive (right) applications

"Delay adaptive" or "rate adaptive" applications: These are typically real time applications such as voice or video over IP. The utility functions that we use for these applications (Fig. 1) are slightly different than those in [17]. In [17], the utility is always strictly positive for non null bandwidth and tends to zero when the bandwidth does. We consider in contrast that the utility equals zero below a certain value, as in [8]. Indeed, in many voice applications, one can select the transmission rate by choosing an appropriate compression mechanism and existing compression software have an upper bound on the compression, which implies a lower bound on the transmission rate for which a communication can be initiated. If there is no sufficient bandwidth, the connection is not initiated. This kind of behavior generates utility functions that are zero for bandwidth below MR and which are not differentiable at the point $(MR, 0)$.

2.2 Fair allocations

Several concepts of fairness are known in the literature: the max-min fairness [3], (as well as the more general concept of weighed max-min fairness) which has been adopted by the ATM-forum [1] for ABR traffic, the proportional fairness [7] the harmonic mean fairness [13], the general fairness criterion that bridges all the above concepts [15] and the Nash Bargaining Solution (NBS).
Nash Bargaining Solution (NBS) Our starting point is the NBP (Nash Bargaining Point) concept [8] for fair allocation, frequently used in cooperative game theory. Let there be n users (or connections). The notion deals directly with fair allocation of achievable utilities of players (and does not require to relate them to the original objects, throughputs in our case, that generate these utilities). Let $U \subset \mathbb{R}^n$ be a closed convex set corresponding to the achievable vectors of utilities of the form $(f_1, ..., f_n)$. Let u_i^0 be a minimum required performance of user i.[1] Let $\mathcal{G} = \{(U, u^0) | U \subset \mathbb{R}^n\}$: it denotes the class of sets of performance measures that satisfy the minimum performance bound u^0 (it contains achievable performances obtained for different utility functions f; in fact, in order to define NBP one has to introduce its performance w.r.t. other utilities, as is seen from property 3 and 5 in the definition below).

[1] In our context, $u_i^0 = f_i(MR_i)$ where f_i is concave increasing. If X is the set of all achievable vectors of bandwidths, then $U = \{f(x) | x \in X\}$.

Definition 2.1. *A mapping $S : \mathcal{G} \to \mathbb{R}^n$ is said to be an NBP (Nash bargaining point) if*

1. *$S(U, u^0) \in U^0 := \{u \in U | u \geq u^0\}$, i.e. it guarantees the minimum required performances.*
2. *It is Pareto optimal* [2]*.*
3. *It is linearly invariant, i.e. the bargaining point is unchanged if the performance objectives are affinely scaled. More precisely, if $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map such that its ith component is given by $\phi_i(u) = a_i u_i + b_i$, then $S(\phi(U), \phi(u^0)) = \phi(S(U, u^0))$.*
4. *S is symmetric i.e. does not depend on the specific labels, i.e. users with the same minimum performance measures and the same utilities will have the same performances.*
5. *S is not affected by enlarging the domain if a solution to the problem with the larger domain can be found on the restricted one. More precisely, if $V \subset U$, $(V, u^0) \in \mathcal{G}$, and $S(U, u^0) \in V$ then $S(U, u^0) = S(V, u^0)$.*

The definition of NBP is thus given through axioms that game theorists find natural to require in seeking for fair assignment. Having defined this concept through the achievable utilities, we define the NBS (Nash Bargaining Solution) in terms of the corresponding strategies (i.e. the allocation of bandwidth that results in the NBP), and then present its characterization through a utility optimization approach.

Definition 2.2. *The point $u^* := S(U, u^0)$ is called the Nash Bargaining Point and $f^{-1}(u^*)$ is called the set of Nash Bargaining Solutions.*

Define $X_0 := \{x \in X | f(x) \geq u^0\}$.

Theorem 2.1. *[8, Thm. 2.1, Thm 2.2]. Let the utility functions f_i be concave, upper-bounded, defined on X which is a convex and compact subset of $\mathcal{L}$. Let J be the set of users able to achieve a performance strictly superior to their initial performance, i.e. $J = \{j \in \{1, ..., n\} | \exists x \in X_0,\ s.t.\ f_j(x) > u_j^0\}$. Assume that $\{f_j\}_{j\in J}$ are injective. Then there exists a unique NBP as well as a unique NBS x that verifies $f_j(x) > u_j(x), j \in J$, and is the unique solution of the problem P_J:*

$$(P_J) \quad \max \prod_{j \in J} (f_j(x) - u_j^0), \quad x \in X_0. \tag{1}$$

Equivalently, it is the unique solution of:

$$(P'_J) \quad \max \sum_{j \in J} \ln(f_j(x) - u_j^0), \quad x \in X_0.$$

Before examining some qualitative implications of the definition, we introduce the very related notion of generalized proportional fairness.

Generalized proportional fairness (GPF). An assignment $x \in X$ is said to be (generalized) proportionally fair with respect to a utility f, if for any other assignment $x^* \in X$, the aggregate of proportional changes in the utilities is zero or negative

$$\sum_{i=0}^{n} \frac{f_i(x_i^*) - f_i(x_i)}{f_i(x_i)} \leq 0. \tag{2}$$

[2] An allocation f is said to be Pareto optimal if it is impossible to strictly increase the allocation of a connection without strictly decreasing another one. The Pareto axiom assures that no bandwidth is "wasted".

Thus, an allocation is GPF if any change in the distribution of the rates would result in the sum of the proportional changes of the utilities to be non-positive. This concept has been defined and applied without considering any utility, i.e. by restricting directly the rates as the object that is assigned fairly [11, 7] (see also [4, 13, 15]). This amounts in taking in (2) $f_i(x_i) = x_i$. Yet, there is no conceptual difference in defining it as we do, i.e. with respect to utilities. In particular, by simply replacing x_i by $f_i(x_i)$, we have the following property (established for the special case $f_i(x_i) = x_i$) of the solution x^{GPF}:

$$x^{GPF} \text{ maximizes } \sum_{i=1}^{n} \ln f_i(x_i) \text{ over } X \tag{3}$$

$$\text{or } x^{GPF} \text{ maximizes } \prod_{i=1}^{n} f_i(x_i) \text{ over } X. \tag{4}$$

The Internet is an example where proportional fairness is used. Indeed, congestion control mechanisms based on linear increase and multiplicative decrease (such as TCP) achieve proportional fairness upper appropriate conditions [7]. The (weighted version of the) proportional fairness is also advocated for future developments of TCP [6].

Comparing with Thm. 2.1, we conclude that GPF coincides with the NBS of [8] when the MR_i's equal zero, and to the original proportional fairness when further restricting to the identity utilities.

We finally note that due to (4) it follows that GPF is invariant under a scale change, i.e. if we multiply the utility f_i of a connection i by a positive constant c_i, the GPF assignment will not change. Yet in general, it will not remain the same under translation by a constant as in NBS.

General fairness criterion. We present another general fairness criterion [15] but apply it to fair allocation of utilities rather than of the rate. Given a positive constant $\alpha \neq 1$, consider the problem

$$\max_x \frac{1}{1-\alpha} \sum_{i=1}^{n} f_i(x_i)^{1-\alpha}, \;\; \alpha \geq 0, \alpha \neq 1. \tag{5}$$

subject to the problem's constraints. This defines a unique allocation which is called the α-bandwidth allocation. This allocation corresponds to the globally optimal allocation as $\alpha \to 0$, to the (generalized) *proportional fairness* when $\alpha \to 1$, to the generalized *harmonic mean fairness* when $\alpha \to 2$, and to the generalized *max-min* allocation when $\alpha \to \infty$ [15]. We shall not deal with this concept until Sec. 6.

2.3 Statement of the general problem

We focus in the paper on the computation of the NBS and briefly compare it to the GPF allocation. Using Thm. 2.1, the NBS is the unique solution $x = x_1, x_2, ..., x_n$ (with n the number of connections) of

$$\begin{aligned} &\max_{x \in X} \prod_{i=1}^{n} (f_i(x_i) - f_i(MR_i)) \text{ where } X_i := \\ &\{x | \forall l = 1, ..., L, (Ax)_l \leq (C)_l, MR_i \leq x_i \leq PR_i\}, \end{aligned} \tag{6}$$

with L the number of links, A the routing matrix (the element $A_{i,j}$ being equal to 1 if connection j goes through link i, 0 otherwise), and C the capacity vector (C_i is the capacity of link i). $(Ax)_l \leq (C)_l$ are the standard capacity constraints. We assume that the network has sufficient bandwidth to satisfy all the users' minimum requirements i.e. $\forall \in 1..L$ we have $\sum_{i=1}^{N} a_{li} MR_i < C_l$.

3 Properties of the NBS and of GPF

As already mentioned, previous references that studied proportional fairness considered the actual bandwidth as the object to be allocated fairly, rather than its utility. The reference [8] who already considered the NBS approach which is defined for general concave utilities, also restricted to linear utilities. Our first goal is thus to study utilities that are more general than those already studied in the following aspects:
(1) allow general concave utilities,
(2) allow $f_i(MR_i)$ to be different from zero.

We note that due to the 3rd axiom in the definition of NBP (Def. 2.1), the second point above will not affect the NBP (and the NBS) but will affect the GPF allocation.

3.1 An example with linear utilities

Consider two connections with the same PR_i and MR_i (we thus omit the index i) that compete over a single link with capacity cap satisfying $2PR > cap > 2MR$. The utility of connection i is $f_i(x) = a_i(x - Z_i)$, $a_i > 0$, $Z_i \leq MR$. Without loss of generality, we assume that $a = a_i$ does not depend on i, since both the NBS as well as GPF are scale invariant. The NBS is clearly $x_i^* = cap/2$, $i = 1, 2$. Define

$$Y_i = \frac{cap}{2} + \frac{Z_i - Z_j}{2}, \quad j \neq i$$

A simple calculation shows that the GPF solution is $x_i^{GPF} = Y_i$ if $Y_i \in [MR_i, PR_i]$, and if not, then for some i, $Y_i < MR_i$. In that case, the GPF solution is $x_i^{GPF} = Y_i$ and $x_j^{GPF} = cap - MR_i$, for $j \neq i$.

This example shows that if we translate the utility of connection i by a positive constant (which implies that Z_i decreases) then its generalized fair share decreases whereas its NBS share does not change.

3.2 The impact of concavity

We now study the impact of concavity on the NBS. Consider two differentiable functions f and g defined on the same interval $[MR, PR]$ where both are strictly positive on $(MR, PR]$. We say that f is more concave than g if for every $x \in (MR, PR]$, the relative derivative of f is smaller than or equal to that of g, i.e. $f'(x)/f(x) \leq g'(x)/g(x)$ (if f or g were not differentiable at x, one could require instead that the same relation holds for the supergradients: if $\hat{f}(x)$ is the largest supergradient of f at x and $\hat{g}(x)$ is the smallest supergradient of g at x, then we require $\hat{f}(x)/f(x) \leq \hat{g}(x)/g(x)$).

Motivated by (4), we say that an assignment x is more fair in the sense of GPF than an assignment y if $\prod_{i=1}^n f_i(x_i) \geq \prod_{i=1}^n f_i(y_i)$. One can define similarly an ordering for the NBS fairness, but then one has to replace f_i by $f_i - f_i(MR_i)$.

In the next example we consider the case in which $f_i(MR_i) = 0$ (so NBS coincides with GPF). Consider 2 connections with utilities f and g as above competing for the bandwidth of a single link. If we had ignored the utilities of the connections, we would have assigned them an equal bandwidth (according to the original proportional allocation), which we denote by $x = cap/2$. We show that by transferring bandwidth from the connection with the more concave utility (say f) to the other one, we improve the fairness (assuming this does not violate the MR and PR constraints) in the sense of GPF or the NBS. Indeed, we have

$$
\begin{aligned}
& g(x+\epsilon)f(x-\epsilon) \\
= \quad & g(x)f(x)\left(1+\epsilon\left[\frac{g'(x)}{g(x)}-\frac{f'(x)}{f(x)}\right]+o(\epsilon)\right)
\end{aligned}
$$

where $o(\epsilon)$ is a function that tends to zero when divided by ϵ as ϵ converges to zero. We conclude that there is some ϵ_0 s.t. for all $\epsilon < \epsilon_0$, $g(x+\epsilon)f(x-\epsilon) > g(x)f(x)$. Hence we strictly improve the fairness by transferring an amount of ϵ_0 to the connection with less concave utility.

By further increasing the amount we transfer, we shall eventually reach a local maximum (since our function is continuous over a compact interval). This will be a global maximum since (4) is a problem of maximization of a concave function over a convex set. We conclude that the fair assignment has the property that more bandwidth is assigned to the less concave function.

Example: Let $f(x) = 3x1\{0 \le x \le 1\} + (2+x)1\{x > 1\}$, and let $g(x) = 2x$ for $x \ge 0$. Then $f'(x)/f(x) = x^{-1}$ for $x \in [0,1)$, and $(2+x)^{-1}$ for $x \ge 1$, whereas $g'(x)/g(x) = x^{-1}$ everywhere. (At $x = 1$, f is not differentiable but its supergradients at that point is the set $[1/3, 1]$). Thus f is more concave than g. We assume that $PR > cap$.

Define $h(x) = 6x(cap - x)$ and $k(x) = 2(x+2)(cap-x)$. The NBS or GPF are obtained as the argument of $\zeta(cap) = \max f(x)g(cap - x)$ which equals

$$
\max\Big(\max_{x\in[0,1]} h(x), \max_{x>1} k(x)\Big).
$$

If $cap > 2$ then $\max_{x\in[0,1]} h(x) = h(1) = 6(cap-1)$, otherwize it is obtained at $x = cap/2$ and equals $3cap^2/2$.
If $cap < 4$ then $\max_{x>1} k(x) = k(1) = 6(cap-1)$, otherwize it is obtained at $1 + cap/2$ and equals $2(1+cap/2)^2$.

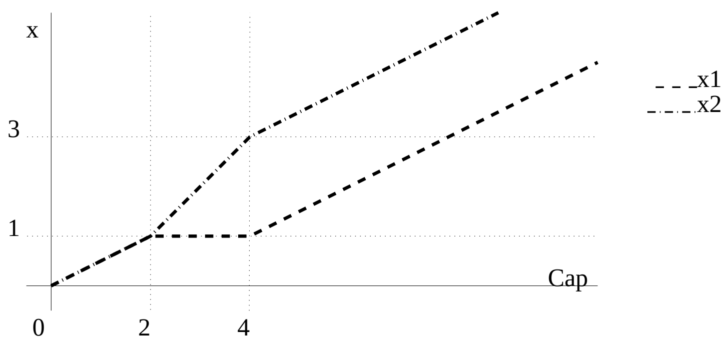

Figure 2: NBS for two connections sharing a link.

The NBS is depicted in Fig 2. We distinguish 3 regions.
(i) $cap < 2$, where $\zeta(cap) = 3cap^2/2$ and the NBS is $x_1^* = cap/2$.
(ii) $2 \le cap < 4$, where $\zeta(cap) = 6(cap-1)$ and $x_1^* = 1$.
(iii) $cap \ge 4$, where $\zeta(cap) = 2(1+cap/2)^2$ and $x_1^* = cap/2 - 1$.

The other connection receives in all cases $x_2^* = cap - x_1^*$. We see in this example that indeed the least concave function receives at least as much as the other one, and the difference increases with cap. It's impressive to note that there is a region in which an increase in the capacity benefits only for one connection. The example illustrates the power of the NBS (or GPF) approach: the original proportional fairness, or even weighted proportional fairness, would assign a proportion of the capacity to each connection that does not vary as we increase the capacity, since it is insensitive to the utilities. In contrast, utility sensitive fairness concepts allocate the bandwidth in a dynamic way: the proportion assigned to each connection is a function of the capacity.

4 Quadratic utility functions

4.1 Definition of the utility function

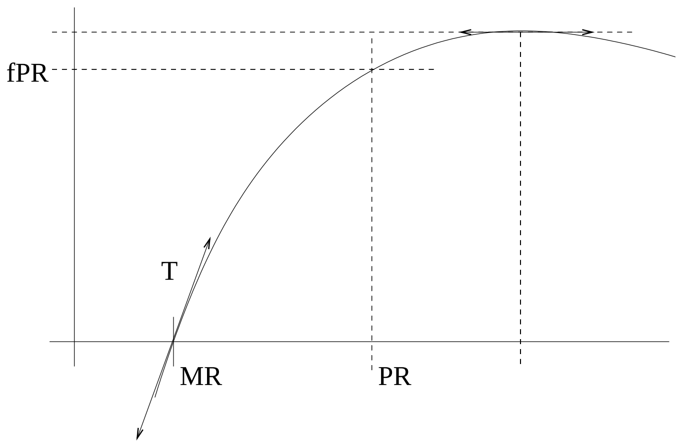

Figure 3: Quadratic utility function.

The utility function of both "elastic traffic" and "delay adaptive" applications have a minimum value MR_i below which it equals zero (in the former case, $MR_i = 0$). As the NBS solution is shift invariant, we can assume without loss of generality that $f_i(MR_i) = 0$. Beyond MR_i the function is concave and increasing with the bandwidth. We can approximate such a utility function with a parabola with several parameters that may depend on the applications (see Fig. 3):
PR_i: max throughput needed by the application
T_i: tangent of the utility function at the point $(MR_i, 0)$
fPR_i: utility value at point PR_i

Note that the utility function is defined only until the point PR_i, so we may ignore the whole right part of the parabola (and in particular, the part in which the function decreases).

As the utility function is a parabola, its general equation has the form: $f_i(x_i) = c_i - a_i(x_i - b_i)^2$. Obviously, f_i can equally be defined by a_i, b_i, c_i or the through the equations $f_i(MR_i) = 0$, $f_i(PR_i) = fPR_i$ and $f_i'(MR_i) = T_i$. We should note that, since PR_i is in the increasing part of the function,

$$\frac{1}{2}T_i(PR_i - MR_i) \leq fPR_i \leq T_i(PR_i - MR_i).$$

We thus define the *concavity of the utility*, β_i through

$$fPR_i = T_i \cdot \beta_i \cdot (PR_i - MR_i)$$

We have: $1/2 \le \beta_i \le 1$ and the smaller β_i is, the more concave is the utility. The limit $\beta_i = 1$ is the linear case (studied in [8]). And therefore:

$$a_i = T_i \frac{1-\beta_i}{PR_i - MR_i},\ b_i = \frac{PR_i - (2\beta_i - 1)MR_i}{2(1-\beta_i)}$$

$$c_i = \frac{T_i}{4}\frac{PR_i - MR_i}{1-\beta_i}.$$

We next present several examples where we use our parabolic utility functions.

4.2 The linear network example

We consider the problem in Fig. 4 in which the squares represent the links and the lines represent the routes. We have $N = L+1$ connections sharing L links. Connection 0 uses all the links, whereas each of the other L connections only goes through a single link (connection i uses link i).

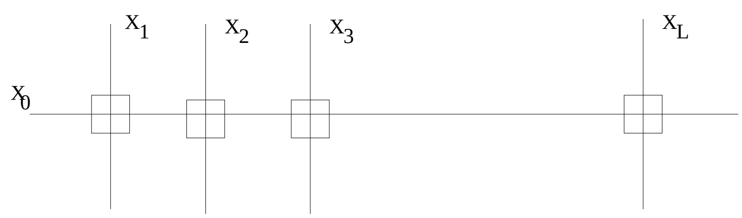

Figure 4: A linear network.

To obtain the NBS, we need to maximize

$$\prod_{i \in \{0,\dots,L\}} f_i(x_i). \tag{7}$$

But, as NBS is Pareto optimal, we have the following constraints for $i = 1, ..., L$: $x_0 + x_i = C_i$ as well as $MR_i \le x_i \le PR_i$. This implies $b_i - \sqrt{\frac{c_i}{a_i}} \le x_i \le b_i$.

We make two significant assumptions. First, that each link has the same capacity cap. Therefore, it is straightforward to notice that each connection i with $1 \le i \le L$ will get the same bandwidth at the equilibrium point. Secondly, we suppose that each of these connections has the same utility function: $\forall i \in \{2, \dots, L\}, a_i = a_1, b_i = b_1, c_i = c_1$.

Therefore the term to maximize in equation (7) becomes: $f_0(x)(f_1(cap - x))^L$ if we denote by x the throughput of the connection x_0.

Solution of the linear problem. By differentiating (7) we then obtain:

$$\begin{aligned} a_0(x - b_0)(c_1 - a_1(cap - x - b_1)^2) = \\ La_1(cap - x - b_1)(c_0 - a_0(x - b_0)^2) \end{aligned} \tag{8}$$

which is a polynomial of the third degree. This can be explicitly solved.

Possible limits. We are interested in the possible limits x_{lim} of the bandwidth assigned to connection x_0 as L grows to infinity.

Lemma 4.1. *Assume $MR_0 + PR_1 \ge cap$. As L grows to infinity, the only possible limit x_{lim} of the bandwidth assigned to connection x_0 is*

$$x^{(3)}_{lim} = b_0 - \sqrt{c_0/a_0} = MR_0, \tag{9}$$

Proof: Equation (8) shows that since x is bounded, the left part of the equation is bounded too. Therefore, the limit x_{lim}, if any, is such that:

$$a_1(cap - x_{lim} - b_1)(c_0 - a_0(x_{lim} - b_0)^2) = 0 \tag{10}$$

Since $MR_0 + PR_1 \geq cap$ and $PR_1 \leq b_1$, we obtain $cap - b_1 < MR_0$ so that the solution is infeasible. The second solution, $x_{lim}^{(2)} = b_0 + \sqrt{c_0/a_0}$ is infeasible as well since we have

$$b_0 + \sqrt{\frac{c_0}{a_0}} = PR_0 + fPR_0 \frac{PR_0 - MR_0}{2 \cdot (1 - \beta_0)}$$

so that this solution is larger than PR_0. The last possible solution is (9), which establishes the proof.

It is interesting to note that the limit x_{lim} does not depend on any parameter of the $i^{th}(i \geq 1)$ connections, or any parameter related to the concavity of the utility function of the connection 0. We show in Fig. 5 how the system converges to the solution as L grows.

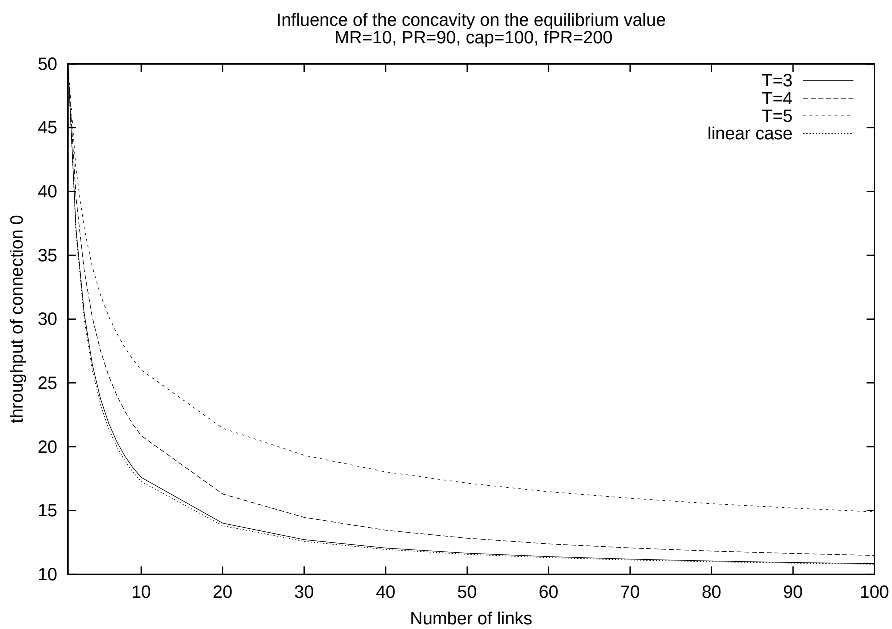

Figure 5: NBS for the linear network.

Remark 4.1. *The condition $MR_0 + PR_1 \geq cap$ in Lemma 4.1 (and in the next propositions) is not restrictive. If it does not hold then we can replace (for any L) MR_0 by $MR_0' := cap - PR_1$ without affecting the NBS, and then apply Lemma 4.1 for MR_0'. Indeed, let x^* be NBS for the original problem. Then $x^* \geq MR_0'$ due to the Pareto optimality of the NBS (2nd element in Definition 2.1). Then it is the NBS for the new problem due to the 5th element in Definition 2.1.*

Asymptotic Analysis. We further refine the analysis of the limit as L becomes large, show that it exists and obtain the rate at which x converges to x_{lim}.

Proposition 4.1. Suppose that $MR_0 + PR_1 \geq cap$, then x verifies:

$$x = MR_0 + Z + o(1/L) \tag{11}$$

with:

$$Z = \frac{cap - MR_0 - MR_1}{2L}\left[1 - \frac{PR_1 - MR_1}{denom}\right],$$

$$\begin{aligned} denom = {} & 2(1-\beta_1)(cap - MR_0) - \\ & PR_1 + MR_1(2\beta_1 - 1) \end{aligned}$$

where $o(1/L)$ is a function that, when divided by L, *tends to zero as L grows to infinity.*

Proof: We shall examine eq. (8) when we substitute $x = x(L)$, which is the solution for a fixed L. As $L \to \infty$, the left hand side tends to a constant:

$$\begin{aligned} \lim_{L\to\infty} a_0(x-b_0)(c_1 - a_1(cap - x - b_1)^2) = {} & \\ -\sqrt{a_0 c_0}\left(c_1 - a_1(cap - b_1 - b_0 + \sqrt{c_0/a_0})^2\right). & \end{aligned} \tag{12}$$

We now examine the right hand side of (8). It can be written as

$$\begin{aligned} & La_1(cap - x - b_1)(c_0 - a_0(x-b_0)^2) \\ = {} & Lf(x)\sqrt{a}(x - MR_0) \end{aligned} \tag{13}$$

where $f(x)$ is given by

$$a_1(cap - x - b_1)(\sqrt{c_0} - \sqrt{a_0}(x - b_0))\sqrt{a_0} \tag{14}$$

and when substituting $x = x(L)$,

$$\lim_{L\to\infty} f(x) = 2a_1(cap - b_1 - b_0 + \sqrt{c_0/a_0})\sqrt{a_0 c_0}. \tag{15}$$

Combining (12)-(15), we conclude that

$$\begin{aligned} & \lim_{L\to\infty} L(x(L) - MR_0) = \\ & -\frac{\sqrt{a_0 c_0}\left(c_1 - a_1(cap - b_1 - b_0 + \sqrt{c_0/a_0})^2\right)}{2a_1(cap - b_1 - b_0 + \sqrt{c_0/a_0})\sqrt{a_0 c_0}} \end{aligned}$$

which yields (11) by substituting the appropriate expressions.

We can notice that:

- the convergence of x is in $1/L$,
- the result does not depend on T_0 nor T_1 (scale invariant),
- in the asymptote, none of the parameters of the 0^{th} connection but MR_0 appears, so that the results are independent of the shape of the utility function of x_0,
- the larger β_1 is, the smaller x gets. This agrees with the conclusions of Subsection 3.2.

In (11), we can easily check the asymptotes for special cases:

- When $\beta_1 \to 1$ we obtain: $Z = \dfrac{cap - MR_0 - MR_1}{L}$ (linear case).
- When $\beta_1 \to 1/2$ we get: $Z =$

$$\frac{cap - MR_0 - MR_1}{2L}\left[1 - \frac{PR_1 - MR_1}{cap - MR_0 - PR_1}\right].$$

4.3 Grid network

This network is the natural generalization of the linear network. It consists of $K \times L$ capacity links with K horizontal routes and L vertical routes as shown in Fig. 6.

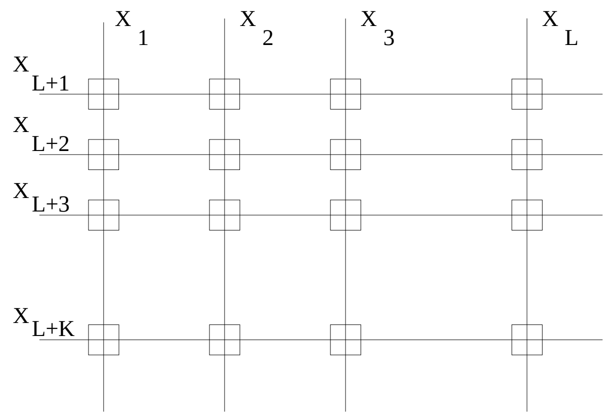

Figure 6: A grid network.

We suppose that all the horizontal connections have the same utility function f_h and each vertical connection has the utility f_v.We can then conclude easily that all the horizontal connections will get the same throughput x and each vertical connection will get the same throughput $x_v = cap - x$.

As in the previous example, we suppose also that, for each $i \in \{1, \dots L\}$,$j \in \{1, \dots K\}$, $MR_i + PR_{L+j} \geq C_{i,L+j}$ and $PR_i + MR_{L+j} \geq C_{i,L+j}$.

We then wish to maximize:

$$\prod_{i \in [0:L]} f_i(x_i) = (f_h(x))^K * (f_v(cap - x))^L. \tag{16}$$

Proposition 4.2. *In the grid network, x verifies: $x = MR_h + Z + o(K/L)$ with:*

$$Z = K \cdot \frac{cap - MR_h - MR_v}{2L} \left[1 - \frac{PR_v - MR_v}{denom} \right]$$

$$\begin{aligned} \textit{with} \quad denom = {} & 2(1 - \beta_v)(cap - MR_h) - \\ & PR_v + MR_v(2\beta_v - 1). \end{aligned}$$

A particular case occurs when $L = K$ and when $f_h = f_v$: we obtain $x = cap/2$.

Proof: This is similar to maximizing: $(f_h(x)) * (f_v(cap - x))^{L/K}$. And then, this problem is equivalent to previous case by substituting L/K instead of L. The second assumption is obvious.

5 Lagrangian method

The Lagrangian method was proposed by [8] to obtain NBS for the special case of linear utility function. It has the advantage of having distributed implementations. We generalize below this approach to the quadratic utility, for which the linear case can be recovered by taking $\beta \to 1$.

5.1 Lagrangian multipliers

We now use the Kuhn Tucker conditions for (6) to obtain alternative characterization of the NBS in terms of the corresponding Lagrange multipliers.

Proposition 5.1. *Under the hypothesis that* $\forall i \in \{1..L\}, \sum a_{li} MR_i < C_l$*, the NBS is characterized by:*
$\exists \mu_l \geq 0, l \in \{1..L\}$ *such that* $\forall i \in \{1..N\}$*, we have*

$$
\begin{aligned}
x_i = \min \Bigg(PR_i, MR_i &+ \left(\sum_{l=1}^{L} \mu_l a_{l,i} \right)^{-1} \\
&+ \frac{1}{2} \cdot \frac{PR_i - MR_i}{1 - \beta_i} \times \\
&\left[1 - \sqrt{1 + \frac{4 \left(\frac{1-\beta_i}{PR_i - MR_i} \right)^2}{\left(\sum_{l=1}^{L} \mu_l a_{l,i} \right)^2}} \right] \Bigg)
\end{aligned}
$$

Proof: Under the assumption $\sum a_{li} MR_i < C_l$, the set A of possible solutions of (6) is non-empty, convex and compact. The constraints in (6) are linear in x_i and $f(x)$ is C^1, therefore the first order Kuhn-Tucker conditions are necessary and sufficient for optimality. The Lagrangian associated with (6) is

$$
\begin{aligned}
\mathcal{L}(x, \lambda, \delta, \mu) &= f(x) - \sum_{i=1}^{n} \lambda_i (MR_i - x_i) \\
&- \sum_{i=1}^{N} \delta_i (x_i - PR_i) - \sum_{l=1}^{L} \mu_l ((Ax)_l - C_l).
\end{aligned}
$$

For $i = 1, ..., n$, $\lambda_i \geq 0$ are the Lagrange multipliers associated with the constraints $x_i \geq MR_i$ and $\delta_i \geq 0$ are those associated with the constraints $x_i \leq PR_i$. $\mu_l \geq 0, l = 1..., L$ are the Lagrange multipliers associated with the capacity constraints. The first order optimality conditions are thus: $\forall i \in \{1, ..., N\}$,

$$
\begin{aligned}
0 &= (\lambda_i - \delta_i - \sum_{l=1}^{L} \mu_l ((Ax)_l - C_l)) \\
&+ \frac{f_i(x)}{f_i(x_i) - f_i(MR_i)} \frac{\partial f_i(x_i)}{\partial x_i}
\end{aligned}
$$

and $\forall i, (x_i - MR_i)\lambda_i = 0$, $(x_i - PR_i)\delta_i = 0$, $\forall l, ((Ax)_l - C_l)\mu_l = 0$. Moreover, $\sum a_{li} MR_i < C_l$ implies that $\forall i, \lambda_i = 0$ as in [8], and either $x_i = PR_i$ or $\delta_i = 0$, which yields the conclusion.

As $\beta \to 1$ we obtain the solution of [8] corresponding to linear utility:

$$
x_i = \min \left(PR_i, MR_i + \left[\sum_{l=1}^{L} \mu_l a_{l,i} \right]^{-1} \right).
$$

μ_l represent the implied cost associated with the network link l. It represents the marginal cost of a rate unit allocated for any connection crossing link l.

5.2 Dual problem

Once we have explicitly expressed the NBS in terms of the Lagrange multipliers, we can actually solve the NBS completely using the dual problem in which we compute the Lagrange multipliers. Define

$$g_i(p) = \begin{cases} PR_i \text{ if } p \leq \dfrac{2\beta_i - 1}{\beta_i}\dfrac{1}{PR_i - MR_i} \\ MR_i + \dfrac{1}{p} + \dfrac{1}{2}\dfrac{PR_i - MR_i}{1-\beta_i} \\ \times \left[1 - \sqrt{1 + \frac{4}{p^2}\cdot\left(\frac{1-\beta_i}{PR_i - MR_i}\right)^2}\right] \\ \qquad \text{otherwise.} \end{cases}$$

$$\text{Then: } x_i(\mu) = g_i(\sum_{l=1}^{L} \mu_l \cdot a_{l,i}). \tag{17}$$

The dual problem is :

$$\begin{gathered} \max_{\mu \in \mathbb{R}_+^L} d(\mu) \\ \text{with } d(\mu) = \min_{x \in X} \mathcal{L}(x,\mu) = \mathcal{L}(\overline{x_i},\mu) \end{gathered} \tag{18}$$

if we note $\overline{x_i}$ the optimal value. The vector $\overline{x} = \overline{x_1, x_2 ... x_n}$ is the NBS. We obtain for each $\mu \in \mathbb{R}^L$:

$$d(\mu) = \sum_{i=1}^{N} \Big[-\ln(f_i(g_i(\sum_{a_{l,i}=1}^{l\in[1..L]} \mu_l)) + \\ \Big(\sum_{a_{l,i}=1}^{l\in[1..L]} \mu_l\Big) g_i\Big(\sum_{a_{l,i}=1}^{l\in[1..L]} \mu_l\Big)\Big] \sum_{l=1}^{L} C_l \mu_l.$$

As in [8], there is no duality gap.

5.3 Decentralized implementation

The dual problem gave an alternative centralized optimization problem for computing the NBS. Still, we can use the decentralized implementation from [8] for the computation, where L local algorithms run at the different nodes. The link updates require information on connections that use that link, and hence global information is not required. The algorithm of [8] for computing μ is: for each $l \in [1..L]$ and $k > 0$, we take:

$$\mu_l^{(k+1)} = \max\left(0, \mu_l^{(k)} + \gamma\left(\sum_{a_{l,i}=1}^{i\in[1..N]} x_i(\mu^{(k)})\right)\right)$$

$$\text{with: } x_i(\mu^{(k)}) = g_i\left(\sum_{l=1}^{L} a_{li}\mu_l^{(k)}\right)$$

and γ a constant step. The initial cost vector $\mu^{(0)} = \mu_1^{(0)}, ..., \mu_L^{(0)}$ is arbitrary and can be chosen equal to zero. Then, [8, Appendix] shows that: $\lim_{k\to\infty} x\left(\mu^{(k)}\right) = \bar{x}$.

6 An SDP solution

In this section, we propose an alternative centralized method for solving the general fairness problem (5), as well as the GPF (3) and NBS (1). It uses a unique mathematical program called semi-definite program, which can be solved in polynomial time in theory and is tractable in practice. The basic idea of SDP is to transform the original maximization problem into a minimization problem of some new variable (or more generally of a linear combination of of variables) subject to a constraint of positive semi-definiteness (psd) [3] of some general matrix P. This matrix is block diagonal, and is thus psd if and only if each block is psd. In our fairness computation, the blocks will always be of size smaller than or equal to two. The psd of the general matrix will (i) imply the capacity constraints as well as those corresponding to the min and max throughputs, (ii) allow us to replace the objective function by a single variable. It thus contains information on the structure of the objective function of the original maximization problem.

SDP involves defining new intermediate variables which are necessary for expressing the required constraints. We sketch in the next subsection some ideas in the construction of the block matrices that are related to different α's (defining the general fairness criterion). We then present a detailed construction of the SDP and in particular, the matrix P and the objective function that will be defined through a scalar product between a vector L and the variables of the SDP. For more details on SDP, see [5]. The program that generates the SDP from the network data is available at http://www-sop.inria.fr/mistral/personnel/Corinne.Touati/. One can then use public domain programs to solve SDP[4].

6.1 Properties of positive semi-definite matrices

Proposition 6.1. *Let w, y and z be three positive real numbers. Then*

$$\begin{pmatrix} w & z \\ z & y \end{pmatrix} \succeq 0 \textit{ if and only if } wy \geq z^2. \tag{19}$$

In particular, if one sets $z = 1$, then the relation $y \geq 1/w$ allows us to obtain constraints of the form $y \geq \sum_{i=1}^{n} {w_i}^{-1}$. This explains how the minimization of $\sum_{i=1}^{n} {w_i}^{-1}$, that appears in the general fairness problem with $\alpha = 2$ is obtained[5] through the minimization of a single variable y subject to the psd matrix constraint in (19).

Thanks to an idea of Nemirovski[16], we can also integrate the following series of functions in our model.

Proposition 6.2. *Let w and y be two real positive number. It is possible, using SDP constraints, to bound w and y by the relation $y \leq w^{k/2^p}$, with $p \in \mathbb{N}$ and $k \in \{0, \ldots, 2^p - 1\}$.*

In other words, if $\alpha \in (0, 1)$ is approximated by some $1 - k/2^p$, then one can generate constraints of the form $y \leq \sum_{i=1}^{n} {w_i}^{1-\alpha}$ and maximizing y is equivalent to maximizing the right member, which solves our problem with very good precision for $0 < \alpha < 1$.

---

[3] A matrix is psd iff its eigenvalues are non-negative. In the case of a matrix of size 2 of the form : $M = \begin{pmatrix} p & r \\ r & q \end{pmatrix}$ with $p \geq 0$ or $q \geq 0$ then M is psd iff $|M| \geq 0$, i.e. $p \cdot q \geq r^2$

[4] see http://www.cs.nyu.edu/cs/faculty/overton/sdppack/sdppack.html

[5] $\alpha = 2$ corresponds to the harmonic fairness that is characterized by the maximizer of $(\sum_i 1/w_i)^{-1}$, or equivalently, the minimizer of $(\sum_i 1/w_i)$, where $w_i = f_i(x_i)$ (another block in the big matrix will take care of the latter equality).

Proof: Let $a_1, \dots, a_p$ be a series of 0/1 integers, such that $k = \sum_{i=1}^{p} a_i 2^{i-1}$. We note $y_0 = 1$, and submit $y_1, \dots, y_p$ to the following constraints:

$$\begin{cases} \begin{pmatrix} y_{i-1} & y_i \\ y_i & w \end{pmatrix} \succeq 0 & \text{if } a_i = 1 \\ \begin{pmatrix} y_{i-1} & y_i \\ y_i & 1 \end{pmatrix} \succeq 0 & \text{if } a_i = 0 \end{cases}$$

Then, obviously, $y_i^2 \leq y_{i-1} w^{a_i}$, and if $y_1, \dots, y_{p-1}$ are submitted to no other constraints, we have:

$$y_p \leq w^{\rho} w^{k/2^p}, \text{ where } \rho := \sum_{i=1}^{p} \frac{a_i}{2^{p+1-i}}$$

Hence the result, by setting $y_p = y$.

Next we present a simple solution for $\alpha > 1$.

Proposition 6.3. *Let w and y be two real positive numbers. It is possible, using SDP constraints, to bound w and y by the relation $y \geq w^{-\beta}$, where $\beta = k/2^p$, $p \in I\!N$ and $k \in \{0, \dots, 2^p - 1\}$.*

This is used to solve the case $\alpha \in (1, 2)$.

Proof: Let z be an intermediate variable. Using proposition 6.2, one can set $z \leq w^{\beta}$. Also one can write

$$\begin{pmatrix} y & 1 \\ 1 & z \end{pmatrix} \succeq 0$$

which leads to $yz \geq 1$. Then w and y are bounded by the unique relation: $yw^{\beta} \geq 1$, hence the result.

Proposition 6.4. *Let w and y be two real positive numbers. It is possible, using SDP constraints, to bound w and y by the relation $y \geq w^{-1/\beta}$, where $\beta = k/2^p$, $p \in I\!N$ and $k \in \{0, \dots, 2^p - 1\}$.*

The proposition covers the cases $\alpha \in (2; +\infty)$.

Proof: Similarly, we obtain $wy^{\beta} \geq 1$.

6.2 Computing the NBS and GPF

The result for the NBS or GPF relies on the following:

Proposition 6.5. *Let y, and $w_1, \dots, w_n$ be real positive numbers. Then using SDP constraints, it is possible to bound these numbers by the relation*

$$y^{2^{\lceil \log_2(n) \rceil}} \leq \prod_{i=1}^{n} w_i.$$

Thus maximizing y leads immediately to the solution of the problems NBS and GPF.

Proof: Let p be the smallest integer such that $2^p \geq n$. We construct a family of real positive variables $y_{i2^k+1,(i+1)2^k}$ with $1 \leq k \leq p$, and $i \in \{0, \dots, 2^{p-k} - 1\}$ satisfying the following constraints:

$$\begin{pmatrix} y_{2i2^{k-1}+1,(2i+1)2^{k-1}} & y_{i2^k+1,(i+1)2^k} \\ y_{i2^k+1,(i+1)2^k} & y_{(2i+1)2^{k-1}+1,(2i+2)2^{k-1}} \end{pmatrix} \succeq 0,$$

where we denote $y_{j,j} = w_j$ for $j \in \{1, \dots, n\}$, $y_{j,j} = 1$ for $j \in \{n+1, \dots, 2^p\}$, and $y = y_{1,2^p}$.

6.3 A simple example

Consider $n = 3$ connections over $L = 4$ links. The connections are defined by the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Element A_{ij} equals 1 if and only if connection j uses link i. In our SDP program, one adds artificial connections so that the total number of connections has the form 2^p with $p \in \mathbb{N}$. (The reason for that will follow from Step 2). As n is not of this form we need to add one extra artificial connection, so that the number of connections is now $n' = 3 + 1 = 4 = 2^2$. We suppose that this extra connection uses its own link, and therefore does not modify the NBS of our problem.

Step 1 The first four blocks of the matrix link the variables x_i with their utility. Therefore, they are of the form:

$$MAT_{1,i} = \begin{pmatrix} -\frac{g_i - c_i}{a_i}, & x_i - b_i \\ x_i - b_i, & 1 \end{pmatrix}$$

We have $MAT_{1,i} \succeq 0 \Leftrightarrow -\frac{g_i - c_i}{a_i} \geq (x_i - b_i)^2 \Leftrightarrow g_i \leq c_i - a_i.(x_i - b_i)^2$. Therefore, maximizing $\prod_i g_i$ will lead to maximizing $\prod_i (c_i - a_i(x_i - b_i)^2)$.

Step 2 The $n' - 1 = 3$ following matrices link the g_i variables together to obtain a single variable that SDP will maximize. These are:

$$\begin{pmatrix} g_1 & g_{12} \\ g_{12} & g_2 \end{pmatrix}, \begin{pmatrix} g_3 & g_{34} \\ g_{34} & g_4 \end{pmatrix}, \begin{pmatrix} g_{12} & g_{1234} \\ g_{1234} & g_{34} \end{pmatrix}$$

The positiveness of these matrices implies that:

$$g_1 \cdot g_2 \geq (g_{12})^2, \; g_3 \cdot g_4 \geq (g_{34})^2$$
$$g_{12} \cdot g_{34} \geq (g_{1234})^2$$

so that $(g_{1234})^4 \leq (g_{12})^2 \cdot (g_{34})^2 \leq g_1 \cdot g_2 \cdot g_3 \cdot g_4$. Then, maximizing the single variable g_{1234} will lead to the required maximization of $\prod_i (c_i - a_i(x_i - b_i)^2)$.

Step 3 We now have to incorporate the linear constraints of the problem: $(Ax)_l \leq cap_l$, $x_i \leq PR_i$, $x_i \geq MR_i$. For this purpose, we add matrices of size 1 (scalar values). SDP will assure us that they are positive (or null) values. Therefore, the constraints $(Ax)_l \leq C_l$ lead to the declaration of L matrices that are in our example:

$$cap_1 - (x_1 + x_3), \quad cap_2 - x_2,$$
$$cap_3 - (x_2 + x_3), \quad cap_4 - x_1.$$

The constraints $x_i \leq PR_i$ and $x_i \geq MR_i$ become in the SDP program 8 matrices of size one that are:

$$\begin{cases} PR_1 - x_1 \\ PR_2 - x_2 \\ PR_3 - x_3 \\ PR_4 - x_4 \end{cases} \quad \begin{cases} x_1 - MR_1 \\ x_2 - MR_2 \\ x_3 - MR_3 \\ x_4 - MR_4 \end{cases}$$

We can notice that the values PR_4 and MR_4 corresponding to the artificial connection are not important since the connection is independent of the others. Whatever the value of PR_4, the solution of SDP will be PR_4. Still, it is important that we bound x_4 otherwize in the programming part it will grow without bound which may cause an error.

The values we should give to the SDP algorithm are the matrix we have just described, plus the vector of variables to minimize. As we want to maximize g_{1234}, we give a negative value to its corresponding coefficient and set $L = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1)$.

Remark 6.1. *In case of n connections with l links, we stress we have at most $6n-5$ variables, $4n-3$ blocks of size 2, and $4n+l-4$ blocks of size 1. Therefore, although our problem is convex, it can be expressed in a simple and short way as a semi-definite program. This is a clear improvement compared to many other specific methods, since the solution can be obtained using any general SDP software[4]. Furthermore, additional conditions (for instance those linked to integer programming, or a weighted optimization on both the bandwidth assignment to connections and the link usage, or other telecommunication-specific requests, such as regulation ones) can now be introduced without further research on convex solving instability and other issues, such as convergence tests for the iterative method, or simply the maintenance of a numerical software.*

6.4 Practical experiments

We implemented the SDP approach using a Matlab program run on a SUN ULTRA 1 computer to obtain the NBS fair share which coincided with GPF (as we took $f_i(MR_i) = 0$). We first tested our program on the same linear network example for which we had explicit expressions for the NBS (Fig. 5), and the results completely agreed.

We then considered two more complex networks which we describe below. The computation time (including the display part) in both cases was less than a minute. In both networks, all links are assumed to have the same capacity (although the program allows to handle different capacities without increasing the complexity). For each network, we present two figures. The first with the set of links and nodes and the second with the set of connections and amount of assigned bandwidth. All connections had the same quadratic utility with the parameters $MR = 10$ and $PR = 80$, $T = 3$, $fPR = 200$. We took $cap = 100$ for all links. Bandwidth parameters and assignments are given in percent of full link capacity.

Consider the network depicted in Fig. 7. It has $L = 10$ links and 11 connections as defined in matrix A below.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The solution given in Fig. 8 involved adding extra 36 intermediate variables, and the matrix involved in the psd constraint was of size 104 (31 block diagonal matrices of size 2, and 42 of size 1).

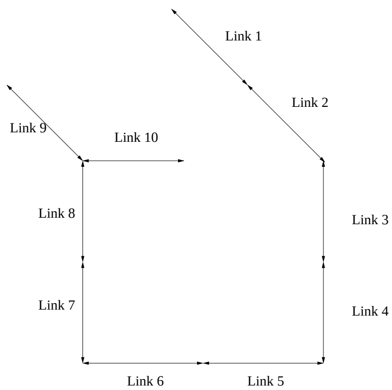

Figure 7: First network: links.

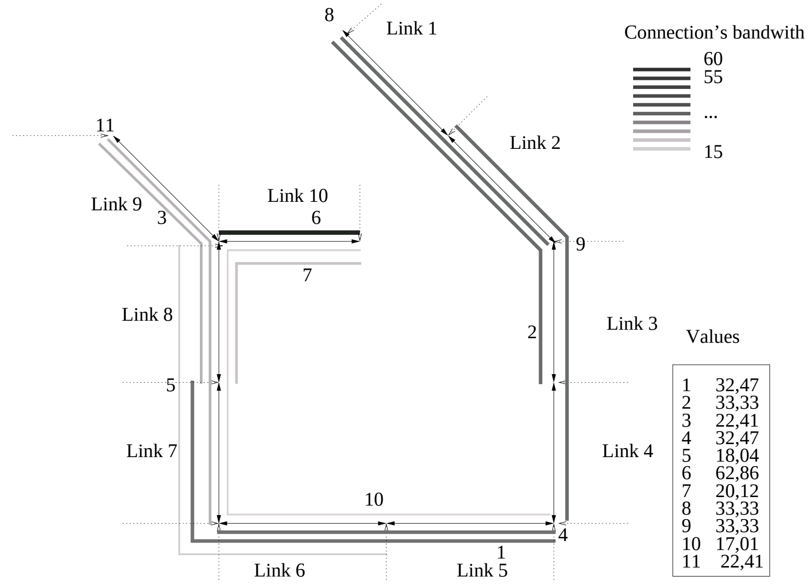

Figure 8: First network: solution.

We considered next the COST experimental network [2], depicted in Fig. 9. It contains 11 nodes, representing the main European capitals. We have considered the 30 connections with the highest forecast demand (We did not include more connections whose forecast demand, based on experiments dating from 1993, were inferior to 2.5 Gb/s). The solution, depicted in Fig. 10 involved adding extra 65 intermediate variables, and the matrix involved in the psd constraint was of size 215 (63 matrices of size 2, and 89 matrices of size 1).

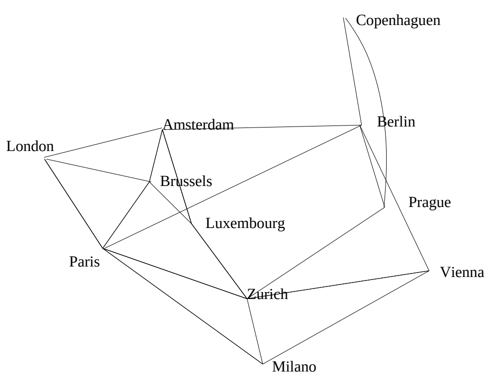

Figure 9: COST network: links.

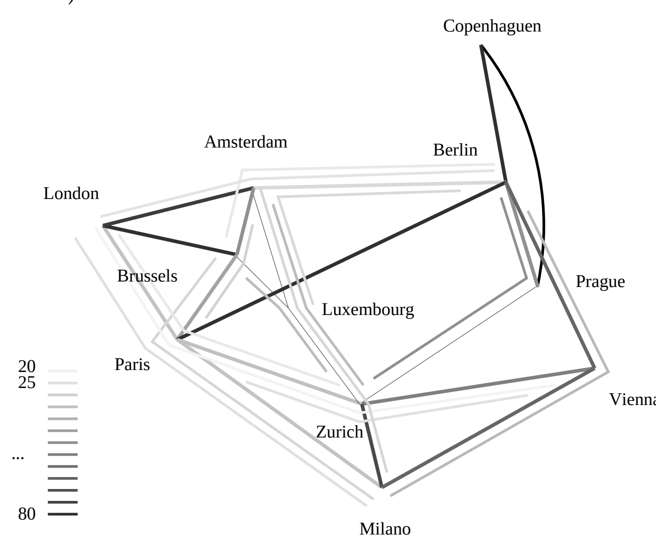

Figure 10: COST network: solution.

Connection	Bandwith	Connection	Bandwith	Connection	Bandwith
London-Paris	33.93	Zurich-Milano	71.58	Milano-Vienna-Berlin	37.00
London-Brussels	80.00	Copenhaguen-Berlin	80.00	Milano-Paris-Brussels	27.93
London-Amsterdam	76.27	Copenhaguen-Prague	80.00	Berlin-Amsterdam-Brussels	22.04
Amsterdam-Berlin	27.11	Berlin-Prague	50.00	Paris-Brussels-Amsterdam	28.42
Amsterdam-Brussels	49.54	Berlin-Vienna	63.00	Paris-Zurich-Vienna	25.48
Brussels-Paris	43.66	Milano-Vienna	63.00	London-Paris-Milano	24.74
Paris-Berlin	80.00	Berlin-Amsterdam-Luxembourg	27.11	London-Paris-Zurich	21.87
Paris-Zurich	33.19	Zurich-Prague-Berlin	50.00	London-Amsterdam-Berlin	23.73
Paris-Milano	47.34	Zurich-Luxembourg-Amsterdam	35.79	Vienna-Zurich-Paris-London	19.46
Zurich-Vienna	55.06	Zurich-Luxembourg-Brussels	35.79	Milano-Zurich-Luxembourg-Amsterdam	28.42

Figure 11: Bandwidth allocation for COST network.

7 Conclusion

We have applied in this paper the NBS approach for bandwidth allocation, as well as the GPF concept that is sensitive to the utilities of connections. concepts that are sensitive to the utilities of connections. We have studied some of the characteristics of these concepts, and showed that they are indeed more suitable for applications that have concave utility. We proposed a simple parameterization of the concavity of the utility function using quadratic functions. We finally proposed some computational approaches that allows us to handle large networks: a Lagrangian approach and a novel approach based on SDP.

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